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Disturbance decoupling using a novel approach to integral sliding-mode

Tobias Posielek[†], Kai Wulff[‡,*] and Johann Reger[‡]

***Abstract*— We propose a novel approach to design an integral sliding mode control (ISMC) for a nonlinear system in regular form. The control method is capable to compensate a class of matched and, in particular, unmatched uncertainties with respect to some given output. Conditions for stability and decoupling are presented. For the case of linear nominal dynamics these conditions take a very simple form. A direct comparison to conventional design methods for ISMC is given and the concepts are illustrated using a simulation example.**

I. Introduction

Sliding mode control is a control technique, widely known for its robustness properties. Disturbances that fulfil a matching condition are completely repelled once the system is in sliding mode. Integral sliding mode (ISM) is a special kind of sliding mode concept that has been introduced by Utkin in [1]. The main feature of this concept is the introduction of an additional integrator state that allows to initialise the system on the sliding manifold and thus eliminate the reaching phase. This establishes the robustness to matched uncertainties throughout the entire system response. This property makes an integral sliding mode controller (ISMC) a common choice for systems with matched external disturbances or model uncertainties.

In [2] an adaptive ISMC is proposed to control the longitudinal rotation of a tilt-rotor aircraft. The ISMC guarantees robustness to bounded matched uncertainties such as sensor noise. An ISMC for a two wheeled-mobile robot is given in [3] to completely annihilate the influence of the joint friction acting on the system.

As the sliding motion is completely insensitive to matched uncertainties only, but not to unmatched ones, minimising the effects of the latter is part of designing an ISMC. One way to deal with unmatched uncertainties in sliding mode control is to find a transformation which results in an integrator chain system which does not contain unmatched uncertainties. However, this requires an observer to estimate the uncertainties [4], [5]. In [6] the integral sliding mode is studied incorporating the effect of the unmatched uncertainties on the reduced dynamics. In [7] a projection matrix is proposed that minimises the effect of the matched disturbance in the reduced dynamics. These results are extended in [8]. In [9] an integral HOSM technique is proposed that utilises the so-called hierarchical quasi-continuous controller design [10]. An extension using a backstepping approach is given in [11].

*Corresponding author: kai.wulff@tu-ilmenau.de

†Author is with the Institute of System Dynamics and Control, German Aerospace Center (DLR), Muenchner Str. 20, D-82234 Wessling, Germany

‡Authors are with the Control Engineering Group, Technische Universität Ilmenau, P.O. Box 10 05 65, D-98684, Ilmenau, Germany

Those conventional ISMC designs choose a nominal control and a projection matrix. This defines the dynamics of the integral state as well as the sliding surface which yields the desired nominal (reduced) dynamics.

The method that we propose chooses the dynamics of the integral state directly in accordance with the design objective and uses the projection matrix to shape the reduced dynamics. This results in reduced dynamics that are typically of lower order than in conventional designs. Furthermore, we derive conditions for which the desired output is completely decoupled from a class of *unmatched* uncertainties.

Similar decoupling approaches have been investigated in [12] and [13]. Both use an input-output linearisation of the system with uncertainties acting only on the internal dynamics. Other approaches utilize an observer to estimate the uncertainty for subsequent compensation, e.g. [14], [15], [16].

The paper is organised as follows. Section II defines the system class and the control objective. In Section III we propose our design method for the ISMC and present stability and decoupling conditions. Moreover, the resulting control is directly compared to the conventional design. Section IV considers the case of linear time-invariant nominal dynamics and derives design conditions for this system class. In Section V we give a simulation example to illustrate and compare the design to conventional approaches.

II. Problem Definition

We consider nonlinear systems of the form

$$\dot{x}_1 = f_1(x_1, x_2) + \phi_1(x_1, x_2) \tag{1a}$$

$$\dot{x}_2 = f_2(x_1, x_2) + b_2(x_1, x_2)u + \phi_2(t, x_1, x_2) \tag{1b}$$

where $x_1(t) \in \mathbb{R}^{n-1}$ and $x_2(t) \in \mathbb{R}$ denote the state, and the control input is $u(t) \in \mathbb{R}$. f_1, f_2 and b_2 are sufficiently smooth vector fields of matching dimensions, where $f_i(0,0) = 0$, $i = 1,2$. The system is subject to the uncertainties $\phi_1 : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and $\phi_2 : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfying some boundary conditions given in the respective sections. To ensure controllability, we require $b_2(x_1, x_2) \neq 0$ for all x_1, x_2. Note that (1) is in regular form [17]. The unmatched uncertainty is denoted as ϕ_1, while ϕ_2 denotes the matched uncertainty. For convenience of notation we introduce the total state $x := (x_1^\top,\ x_2)^\top$ as well as $f(x) := (f_1(x_1, x_2)^\top,\ f_2(x_1, x_2))^\top$, $b(x) := (0,\ b_2(x_1, x_2))^\top$, and the overall uncertainty $\phi := (\phi_1^\top,\ \phi_2)^\top$.

The control objective is to guarantee the existence of an asymptotically stable equilibrium of the closed-loop system and

$$\lim_{t\to\infty} h(x_1(t), x_2(t)) = 0 \tag{2}$$

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for some uniformly continuous output function h and unknown ϕ_1 and ϕ_2. Furthermore, the matched disturbance ϕ_2 shall be compensated on h for all $t \geq 0$. Moreover, we shall derive conditions for which a part of the *unmatched disturbance* ϕ_1 is completely compensated on h for all $t \geq 0$.

III. Proposed Integral Sliding-Mode Design

In this section we propose an integral sliding mode controller that solves the control task. While the discontinuous part of the sliding mode control ensures robustness against the matched uncertainty, the designed integral state guarantees a vanishing stationary error of the desired output h, as in (2), when *unmatched* uncertainties are present.

Define the integrator state z as per

$$\dot{z} = h(x_1, x_2) \tag{3}$$

in view of (2). We choose the switching function

$$s(x_1, x_2, z) = g(x_1, x_2) + z \tag{4}$$

where g is to be selected such that $\frac{\partial g(x_1,x_2)}{\partial x_2} \neq 0$ for all $(x_1, x_2) \in \mathbb{R}^n$. Then there is a function $l : \mathbb{R}^n \to \mathbb{R}$ ensuring

$$s(x_1, x_2, z) = 0 \Leftrightarrow x_2 = l(x_1, z)\,. \tag{5}$$

In order to obtain a more compact notation, we define

$$G(x) := \begin{bmatrix} G_1(x)^\top & G_2(x) \end{bmatrix} := \begin{bmatrix} \frac{\partial g(x_1,x_2)}{\partial x_1} & \frac{\partial g(x_1,x_2)}{\partial x_2} \end{bmatrix}$$

where $G_1(x) \in \mathbb{R}^{n-1}$ and $G_2(x) \in \mathbb{R}$. Then the control law that yields $\dot{s} = -\rho\,\mathrm{sgn}(s)$ for $\phi \equiv 0$ and $\rho > 0$ is given by

$$u(x) = \frac{-1}{G_2(x) b_2(x)} \left(G(x) f(x) + h(x) + \rho\,\mathrm{sgn}(s) \right). \tag{6}$$

Choosing the initial state $z(0)$ according to

$$z(0) = -g(x_1(0), x_2(0)) \tag{7}$$

guarantees sliding-mode for $t = 0$. The reduced dynamics including the uncertainties then takes the shape

$$\dot{x}_1 = f_1(x_1, l(x_1, z)) + \phi_1(x_1, x_2) \tag{8a}$$
$$\dot{z} = h(x_1, l(x_1, z))\,. \tag{8b}$$

A. Stability analysis

In this section we shall derive a sufficient condition for stability of the proposed approach. In order to formulate our result, we shall note a number of assumptions in terms of the vector $w := (x_1^\top, z)^\top \in \mathbb{R}^n$.

For $\phi_1(0) \neq 0$ the origin is not an equilibrium point of the reduced dynamics (8). Thus, we require

Assumption 1 *There exists an equilibrium point $w^* := ({x_1^*}^\top, z^*)^\top$ with $x_2^* = l(x_1^*, z^*)$ such that*

$$f_1(x_1^*, l(x_1^*, z^*)) + \phi_1(x_1^*, l(x_1^*, z^*)) = 0\,, \tag{9a}$$
$$h(x_1^*, l(x_1^*, z^*)) = 0\,. \tag{9b}$$

We further require that (8) meets linear growth bounds:

Assumption 2 *There exists a neighbourhood W^* of the origin such that for all $w^* \in W^*$ there is a neighbourhood W of $({x_1^*}^\top, z^*)^\top$ where for all $w \in W$ holds:*

$$\begin{aligned} \| f_1(x_1, l(x_1, z)) - f_1\big(x_1^*, l(x_1^*, z^*)\big) \\ - f_1(\Delta x_1, l(\Delta x_1, \Delta z)) \| \leq \gamma_1 \|\Delta w\|\,, \end{aligned} \tag{10}$$

$$\| h\,(x_1, l(x_1, z)) - h\,(\Delta x_1, l(\Delta x_1, \Delta z)) \| \leq \gamma_3 \|\Delta w\|\,. \tag{11}$$

with $\Delta x_1 = x_1 - x_1^*$, $\Delta z = z - z^*$ and $\Delta w = w - w^*$.

Furthermore, g in switching function (4) has to be chosen such that the origin of the reduced dynamics (8) is asymptotically stable in the absence of uncertainties, i.e. $\phi_1 \equiv 0$:

Assumption 3 *There exists a Lyapunov function V that for all Δw with $w^* \in W^*$ and $w \in W$ satisfies the conditions*

$$c_1 \|\Delta w\|^2 \leq V(\Delta w) \leq c_2 \|\Delta w\|^2\,, \tag{12a}$$

$$\frac{\partial V(\Delta w)}{\partial \Delta w} \begin{bmatrix} f_1(\Delta x_1, l(\Delta x_1, \Delta z)) \\ h(\Delta x_1, l(\Delta x_1, \Delta z)) \end{bmatrix} \leq -c_3 \|\Delta w\|^2\,, \tag{12b}$$

$$\left\| \frac{\partial V(\Delta w)}{\partial \Delta w} \right\| \leq c_4 \|\Delta w\|\,. \tag{12c}$$

Note that Assumption 3 leads to uniform asymptotic stability of the closed loop system. Relaxing this stability requirements may lead to less restrictive conditions.

Finally, we have some requirements for the overall uncertainty $\phi := (\phi_1^\top,\ \phi_2)^\top$ and the unmatched part ϕ_1.

Assumption 4 *The uncertainty $\phi(x)$ projected by $G(x)$ is locally bounded by $\phi^{\sup} > 0$, i.e. there exists an invariant set $\mathbb{X} \subseteq \mathbb{R}^n$ such that for all $x \in \mathbb{X}$ holds:*

$$\|G(x)\phi(t, x)\| \leq \phi^{\sup}\,. \tag{13}$$

Further, the unmatched uncertainty ϕ_1 shall satisfy a linear growth bound $\gamma_2 > 0$ on the sliding manifold for all $w^* \in W^*$ and $w \in W$:

$$\|\phi_1\big(x_1, l(x_1, z)\big) - \phi_1\big(x_1^*, l(x_1^*, z^*)\big)\| \leq \gamma_2 \|\Delta w\|\,. \tag{14}$$

Note that condition (13) is often required to hold for all $x \in \mathbb{R}^n$. However, this may turn out too restrictive for many practical applications and can be relaxed if an invariant subset $\mathbb{X} \subseteq \mathbb{R}^n$ exists as stated in our assumption.

Theorem 1 *Consider system* (1) *augmented by* (3) *with the integral sliding mode control law* (6) *subject to the Assumptions 1 to 4. Then the sliding manifold $s(x_1, x_2, z) = 0$ is reached in finite time if $\rho \geq \phi^{\sup} + \varepsilon$ for $\varepsilon > 0$, $x(0) \in \mathbb{X}$.*

Proof: We apply a standard technique for showing this property and take a Lyapunov function with $\dot{V} \leq -\alpha V^{\frac{1}{2}}$, $\alpha > 0$. This gives finite time convergence to $s = 0$ [18]. Consider $V(s) = \frac{1}{2}s^2$. It follows

$$\begin{aligned} \dot{V} = s\dot{s} &= s\big(G(x)\dot{x} + h(x)\big) \\ &= s\big(G_1(x) f_1(x) + G_2(x) f_2(x) \\ &\quad + G_2(x) b_2(x) u(x) + G(x)\phi(x) + h(x)\big)\,. \end{aligned}$$

Substituting the control law (6) we obtain

$$\dot{V} = s\big(G(x)\phi(x) - \rho\,\mathrm{sgn}(s)\big) \leq \|G(x)\phi(x)\|\|s\| - \rho\|s\| \\ \leq -\varepsilon\|s\|\,. \qquad (15)$$

■

Then we can state the main theorem of this section.

Theorem 2 *Consider system* (1) *augmented by* (3) *with the integral sliding mode control law* (6) *and initialisation* (7). *Let Assumptions 1– 4 be fulfilled with* $\sqrt{2}(\gamma_1+\gamma_2+\gamma_3) < \frac{c_3}{c_4}$. *Then in the closed loop system the equilibrium point* (x_1^*, x_2^*, z^*) *is locally asymptotically stable, the control objective* (2) *is achieved, and the matched uncertainty is compensated completely.*

Proof: In order to show stability of the closed loop dynamics, it is sufficient to show stability of the reduced dynamics as the choice of $z(0) = -g(x_1(0), x_2(0))$ and condition (15) ensures the initialisation and existence of the sliding mode for all times. We write the reduced dynamics (8) in relative coordinates $\Delta x_1, \Delta x_2$ and Δz, that is

$$\begin{aligned}\Delta\dot{x}_1 &= f_1(x_1^* + \Delta x_1, l(x_1^* + \Delta x_1, z^* + \Delta z)) \\ &\quad + \phi_1(x_1^* + \Delta x_1, l(x_1^* + \Delta x_1, z^* + \Delta z))\,, \\ \Delta\dot{z} &= h(x_1^* + \Delta x_1, l(x_1^* + \Delta x_1, z^* + \Delta z))\,.\end{aligned}$$

Defining the perturbation $\zeta = \begin{bmatrix}\zeta_1^\top & \zeta_2\end{bmatrix}$ with

$$\begin{aligned}\zeta_1(\Delta x_1, \Delta z) &:= f_1\big(x_1, l(x_1, z)\big) - f_1\big(\Delta x_1, l(\Delta x_1, \Delta z)\big) \\ &\quad + \phi_1(x_1, l(x_1, z))\,, \\ \zeta_2(\Delta x_1, \Delta z) &:= h(x_1, l(x_1, z)) - h(\Delta x_1, l(\Delta x_1, \Delta z))\,.\end{aligned}$$

we obtain the perturbed system

$$\Delta\dot{x}_1 = f_1(\Delta x_1, l(\Delta x_1, \Delta z)) + \zeta_1(\Delta x_1, \Delta z)\,, \qquad (16a)$$

$$\Delta\dot{z} = h(\Delta x_1, l(\Delta x_1, \Delta z)) + \zeta_2(\Delta x_1, \Delta z)\,. \qquad (16b)$$

Substituting (10), (11) and (14) yields

$$\begin{aligned}\|\zeta_1(\Delta x_1, \Delta z)\| &= \|f_1\big(x_1, l(x_1, z)\big) - f_1\big(\Delta x_1, l(\Delta x_1, \Delta z)\big) \\ &\quad - f_1\big(x_1^*, l(x_1^*, z^*)\big) - \phi_1(x_1^*, l(x_1^*, z^*)) \\ &\quad + \phi_1(x_1, l(x_1, z))\| \\ &\leq \|f_1\big(x_1, l(x_1, z)\big) - f_1\big(\Delta x_1, l(\Delta x_1, \Delta z)\big) \\ &\quad - f_1\big(x_1^*, l(x_1^*, z^*)\big)\| \\ &\quad + \|\phi_1\big(x_1, l(x_1, z)\big) - \phi_1\big(x_1^*, l(x_1^*, z^*)\big)\| \\ &\leq (\gamma_1 + \gamma_2)\|\Delta w\|\,, \\ \|\zeta_2(\Delta x_1, \Delta z)\| &= \|h\big(x_1, l(x_1, z)\big) - h\big(\Delta x_1, l(\Delta x_1, \Delta z)\big)\| \\ &\leq \gamma_3\|\Delta w\|\,.\end{aligned}$$

Using the Euclidean norm and Hölder's inequality we obtain

$$\begin{aligned}\|\zeta(\Delta x_1, \Delta z)\| &\leq \sqrt{2}(\|\zeta_1(\Delta x_1, \Delta z)\| + \|\zeta_2(\Delta x_1, \Delta z)\|) \\ &\leq \sqrt{2}(\gamma_1 + \gamma_2 + \gamma_3)\|\Delta w\|\,. \qquad (17)\end{aligned}$$

Consider the Lyapunov function V of Assumption 3 for system (16b) and take its time derivative

$$\begin{aligned}\dot{V}(\Delta w) &= \frac{\partial V}{\partial \Delta w}\left(\begin{bmatrix} f_1(\Delta x_1, l(\Delta x_1, \Delta z)) \\ h(\Delta x_1, l(\Delta x_1, \Delta z))\end{bmatrix} + \zeta(\Delta w)\right) \\ &\leq \frac{\partial V}{\partial \Delta w}\begin{bmatrix} f_1(\Delta x_1, l(\Delta x_1, \Delta z)) \\ h(\Delta x_1, l(\Delta x_1, \Delta z))\end{bmatrix} + \left\|\frac{\partial V}{\partial \Delta w}\zeta(\Delta w)\right\|.\end{aligned}$$

With (12) and (17) we obtain

$$\dot{V}(\Delta w) \leq -c_3\|\Delta w\|^2 + c_4\sqrt{2}(\gamma_1 + \gamma_2 + \gamma_3)\|\Delta w\|^2\,.$$

Thus $\dot{V}(\Delta w)$ is strictly negative if $\sqrt{2}(\gamma_1 + \gamma_2 + \gamma_3) < \frac{c_3}{c_4}$.

This renders the equilibrium point (x_1^*, z^*) of system (8) asymptotically stable which implies that $\lim_{t\to\infty} z$ exist. Hence, in view of (3) the time integral of h takes a limit. Since h is uniformly continuous, with Barbălat's Lemma we conclude that $\lim_{t\to\infty} h(x_1, x_2) = 0$. Furthermore, due to the initialisation (7) the closed-loop system is in sliding-mode for $t \geq 0$ and thus the matched uncertainty ϕ_2 is compensated completely. ■

Remark 3 *In [6] similar arguments are used as in Theorem 2 to show stability of a conventional integral sliding mode controller.*

B. Comparison to conventional integral sliding mode

In conventional integral sliding mode as proposed in [1] with several design propositions, e.g. [7],[19],[8], a sliding manifold and integral state is designed based on a conventionally designed nominal control u_0 where

$$\dot{z} = -G(x)f(x) - G_2(x)b_2(x)u_0(x)\,, \qquad (18)$$

$$s(x_1, x_2, z) = g(x_1, x_2) + z\,. \qquad (19)$$

Then, the reduced dynamics are the same as in the nominal case but without the influence of the matched uncertainty. Unmatched uncertainties can be compensated to the extent of the nominal control u_0.

If we cast our approach into this scheme then we can identify the continuous nominal control u_0 and the discontinuous control u_1 such that $u = u_0 + u_1$ in terms of

$$\begin{aligned}u_0(x) &= \frac{-1}{G_2(x)b_2(x)}(G(x)f(x) + h(x))\,, \\ u_1(x) &= \frac{-1}{G_2(x)b_2(x)}\rho\,\mathrm{sgn}(s)\,.\end{aligned}$$

An overview of the two different design methods for an integral sliding mode controller is given by Table I.

In the conventional design all effort has to be put into the choice of the nominal control u_0 to achieve the control objective (2). In particular, the integrator state is determined by the choice of u_0 and is not chosen, as in our proposal, directly to guarantee the control objective. Furthermore, the choice of g in the switching function does not influence the reduced dynamics in (x_1, x_2). In the proposed method g influences the reduced dynamics significantly and thus allows to shape the desired dynamics.

Furthermore, the order of the reduced dynamics in the conventional design is n plus the order needed for the nominal controller. Our approach results in reduced dynamics of order n. Thus, assuming a nominal control with integral action for the conventional design, our proposed approach will always yield reduced dynamics of lower order. This may lead to a simpler stability analysis and less restrictive requirements.

Note also, since our proposed method can be cast into the standard framework, results obtained for integral sliding mode also extend to our method. For example, the choice of $G(x)^\top = \begin{bmatrix} 0 & b_2(x) \end{bmatrix}$ minimises the effect of an unmatched disturbance in the sense of [8].

C. Decoupling of unmatched disturbances

In this section, we consider a special case of (1) and show that the switching function can be designed such that at least some components of the unmatched disturbance are fully compensated for all $t \geq 0$ if $h(x_1, x_2) = h^{\mathrm{d}}(x_{11})$ depending only on some component of x_1. Let $x_1(t)$ be split into $x_{11}(t) \in \mathbb{R}^{n-2}$ and $x_{12}(t) \in \mathbb{R}$ such that system (1) reads

$$\dot{x}_{11} = f_{11}(x_{11}, x_{12}, x_2) + \phi_{11}(x_{11}) \tag{20a}$$
$$\dot{x}_{12} = f_{12}(x_{11}, x_{12}, x_2) + \phi_{12}(x_{11}, x_{12}, x_2) \tag{20b}$$
$$\dot{x}_2 = f_2(x_1, x_2) + b_2(x_1, x_2)u + \phi_2(t, x_1, x_2)\,. \tag{20c}$$

Theorem 4 *Consider system* (20) *augmented by* (3) *with the integral sliding mode control law* (6) *and initialisation* (7). *Let Assumptions 1– 4 be fulfilled with* $\sqrt{2}(\gamma_1+\gamma_2+\gamma_3) < \frac{c_3}{c_4}$.

Choose the integrator state (3) *such that*

$$\dot{z} = h^{\mathrm{d}}(x_{11}) \tag{21}$$

and the switching function (4) *such that the first component of the reduced dynamics* (8) *is independent of* x_{12}*, i.e.*

$$f_{11}^{\mathrm{d}}(x_{11}, z) := f_{11}(x_{11}, x_{12}, l(x_{11}, x_{12}, z))\,. \tag{22}$$

Then in the closed loop system the equilibrium point (x_1^*, x_2^*, z^*) *is locally asymptotically stable, the control objective* (2) *is fulfilled, the matched uncertainty* ϕ_2*, and the unmatched uncertainty* ϕ_{12} *are compensated completely on* h *for all* $t \geq 0$.

Proof: The asymptotic stability directly follows from Theorem 2. For the reduced dynamics (8) with (20) and (22) we have

$$\dot{x}_{11} = f_{11}^{\mathrm{d}}(x_{11}, z) + \phi_{11}(x_{11}) \tag{23a}$$
$$\begin{aligned}\dot{x}_{12} = {}& f_{12}(x_{11}, x_{12}, l(x_{11}, x_{12}, z)) \\ &+ \phi_{12}(x_{11}, x_{12}, l(x_{11}, x_{12}, z))\end{aligned} \tag{23b}$$
$$\dot{z} = h^{\mathrm{d}}(x_{11})\,. \tag{23c}$$

From the reduced dynamics of x_{11} and z in (23a) and (23c) we can see that the disturbance ϕ_{12} does not influence the dynamics of x_{11}, neither directly nor via the influence of states disturbed by ϕ_{12}. ■

Remark 5 *If* (23) *is bounded-input bounded-state stable for input* ϕ_{12} *and state* x_{12} *then we can also allow that* ϕ_{12} *shows an additional time-varying component, e.g. an external disturbance input, without violating control objective* (2).

Remark 6 *The decoupling conditions* (21), (22) *can also be achieved using other (non-integral) sliding-mode techniques as shown in the numerical example in Section V.*

IV. Special case: linear system

In this section we shall consider the linear case and apply our proposed ISM design. Consider system (1) with

$$\begin{aligned}\dot{x}_{11} &= A_{11}x_{11} + A_{12}x_{12} + A_{13}x_2 + \phi_{11}(x_{11}) \\ \dot{x}_{12} &= A_{21}x_{11} + A_{22}x_{12} + A_{23}x_2 + \phi_{12}(x_{11}, x_{12}, x_2) \\ \dot{x}_2 &= A_{31}x_{11} + A_{32}x_{12} + A_{33}x_2 + u + \phi_2(t, x_{11}, x_{12}, x_2)\end{aligned}$$

where $x_{11}(t) \in \mathbb{R}^{n-2}$ and $x_{12}(t), x_2(t), u(t) \in \mathbb{R}$. Note that this type of system results generically from a time-invariant nonlinear system where only some linear part is certain and the nonlinearities are considered as uncertainties ϕ_i.

The output of interest in view of (2) is

$$y = h(x) = \begin{bmatrix} H_{11} & H_{12} & H_2 \end{bmatrix} x.$$

A. Proposed ISM with disturbance decoupling

We choose the integral state z to comply with

$$\dot{z} = h(x_1, x_2) = H_{11}x_{11} + H_{12}x_{12} + H_2x_2 \tag{24}$$

and select the switching function

$$s = G_{11}x_{11} + G_{12}x_{12} + G_2x_2 + z \tag{25}$$

with $G_{11} \in \mathbb{R}^{n-2}$ and $G_{12}, G_2 \in \mathbb{R}$ such that the sliding manifold $s \equiv 0$ is given by

$$x_2 = l(x_{11}, x_{12}, z) = -\frac{G_{11}}{G_2}x_{11} - \frac{G_{12}}{G_2}x_{12} - \frac{1}{G_2}z\,. \tag{26}$$

We obtain the reduced dynamics

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{z} \end{bmatrix} = A^{\mathrm{red}} \begin{bmatrix} x_{11} \\ x_{12} \\ z \end{bmatrix} + \begin{bmatrix} \phi_{11}(x_1) \\ \phi_{12}(x_{11}, x_{12}, l(x_{11}, x_{12}, z)) \\ 0 \end{bmatrix}, \tag{27}$$

with

$$A^{\mathrm{red}} = \begin{bmatrix} A_{11} - A_{13}\frac{G_{11}}{G_2} & A_{12} - A_{13}\frac{G_{12}}{G_2} & -A_{13}\frac{1}{G_2} \\ A_{21} - A_{23}\frac{G_{11}}{G_2} & A_{22} - A_{23}\frac{G_{12}}{G_2} & -A_{23}\frac{1}{G_2} \\ H_{11} - H_2\frac{G_{11}}{G_2} & H_{12} - H_2\frac{G_{12}}{G_2} & -H_2\frac{1}{G_2} \end{bmatrix}.$$

In view of conditions (21) and (22) of Theorem 4 we may decouple the uncertainty ϕ_{12} from the desired output h if

$$A_{13}\frac{G_{12}}{G_2} = A_{12}\,, \quad H_{12} = 0\,, \quad H_2 = 0\,. \tag{28}$$

The remaining parameters G_{11}, G_2 can be used to shape the dynamics of the desired output Hx. Substituting (28) into (27), we are left to design the state feedback

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ H_{11} & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ z \end{bmatrix} - \begin{bmatrix} A_{13} \\ 0 \end{bmatrix} K \begin{bmatrix} x_{11} \\ z \end{bmatrix} \tag{29}$$

	Conventional ISMC design method	Proposed ISMC design method
Design parameter	Nominal Control u_0	Dynamic of the integrator state h
	Switching function via g	Switching function via g
Switching function	$s(x_1,x_2,z)=g(x_1,x_2)+z$	$s(x_1,x_2,z)=g(x_1,x_2)+z$
	$s(x_1,x_2,z)=0 \Leftrightarrow x_2=l(x_1,z)$	$s(x_1,x_2,z)=0 \Leftrightarrow x_2=l(x_1,z)$
Control law	$u=u_0+u_1$	$u=u_0+u_1$
		$u_0=-[G_2(x)b_2(x)]^{-1}(G(x)f(x)+h(x))$
	$u_1=-[G_2(x)b_2(x)]^{-1}\rho\,\mathrm{sgn}(s)$	$u_1=-[G_2(x)b_2(x)]^{-1}\rho\,\mathrm{sgn}(s)$
Integrator state	$\dot z=-G(x)f(x)-G_2(x)b_2(x)u_0(x)$	$\dot z=h(x_1,x_2)$
Red. dyn. in (x_1,x_2)	$\dot x_1=f_1(x)+\phi_1(x)$	$\dot x_1=f_1(x)+\phi_1(x)$
	$\dot x_2=f_2(x)+b_2(x)u_0(x)-b_2(x)G_1(x)\phi_1(x)$	$\dot x_2=-[G_2(x)]^{-1}\big(G_1(x)f_1(x)-h(x)\big)-b_2(x)G_1(x)\phi_1(x)$
Red. dyn. in (x_1,z)	$\dot x_1=f_1(x_1,l(x_1,z))+\phi_1(x_1,l(x_1,z)$	$\dot x_1=f_1(x_1,l(x_1,z))+\phi_1(x_1,l(x_1,z))$
$(w=(x_1,l(x_1,z)))$	$\dot z=-G(w)f(w)-G_2(w)b_2(w)u_0(w)$	$\dot z=h(x_1,l(x_1,z))$
Condition on ρ	$\rho>\lVert G_1\phi_1+G_2\phi_2\rVert$	$\rho>\lVert G_1\phi_1+G_2\phi_2\rVert$

TABLE I

COMPARISON OF ISMC DESIGN METHODS FOR SYSTEM (1).

with $K=\begin{bmatrix}\frac{G_{11}}{G_2} & \frac{1}{G_2}\end{bmatrix}$. Additionally, stability of x_{12} shall be guaranteed in the nominal case $\phi_1\equiv 0$, which results in

$$A_{22}-A_{23}\frac{G_{12}}{G_2}<0\,. \tag{30}$$

Thus, the design (29) is to meet the constraints (28) and (30).

It remains to check Assumptions 1-4 of Theorem 2. Assumption 1 requires a stationary solution. This solution is trivial if the perturbation is vanishing at the origin, i.e. $\phi(0)=0$. Assumption 2 is fulfilled since we deal with a linear system. Assumption 3 is fulfilled by the choice of K in (29) and condition (30). Assumption 4 needs to be checked with some knowledge about the uncertainty. We may then choose $\rho>\phi^{\sup}$ as in Theorem 1 to ensure the existence of the sliding mode.

V. SIMULATION EXAMPLE

In this section we illustrate the proposed design with an example and compare the results to conventional ISM design, [7], [19], [8]. In the spirit of the example in [19] we consider the following system with $x=\begin{bmatrix}x_{11} & x_{12} & x_2\end{bmatrix}^\top\in\mathbb{R}^3$:

$$\dot x_{11}=x_{12}+x_2+\phi_{11}(x_{11}) \tag{31a}$$
$$\dot x_{12}=x_2+\phi_{12}(t,x) \tag{31b}$$
$$\dot x_2=f_2(t,x)+b_2(t,x)u+\phi_2(t,x) \tag{31c}$$

with

$$f_2(t,x)=x_{11}x_{12}-x_2\sin(x_{11})+\cos(t)$$
$$b_2(t,x)=3+\frac{2}{\pi}\arctan(x_2)+0.5\sin(t)\,.$$

Note that we inherit the time-varying uncertainty ϕ_{12} from the example in [19]. Such uncertainty is not covered by Theorem 4. However, the example shows that the approach is appropriate for time-varying uncertainty in some cases.

The control objective is to obtain an asymptotically stable equilibrium and $\lim_{t\to\infty}h(x)=0$ with desired output $h(x)=x_{11}$. We use our proposed method to design an ISMC which achieves the control objective and also decouples two of three process states from the uncertainty. For comparison, we take two conventionally designed ISMCs. For the nominal control we consider a PI state-feedback that achieves the control objective as well as a sliding mode control law satisfying (21) for decoupling. The sliding manifold is chosen according to [7] such that the discontinuous part does not amplify the effect of the unmatched uncertainties ϕ_{11}, ϕ_{12}.

A. Proposed ISM design

According to (3) and (7), we choose:

$$\dot z=Hx=\begin{bmatrix}1 & 0 & 0\end{bmatrix}x\,,\quad z(0)=-Gx(0)$$

and the switching function shall be parametrised as in (25) with sliding-manifold (26). From the decoupling condition (28), we get the requirement $\frac{G_{12}}{G_2}=1$. This also satisfies the stability requirement (30) for the internal state x_{12}.

For the remaining dynamics we consider (29)

$$\begin{bmatrix}\dot x_{11}\\ \dot z\end{bmatrix}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_{11}\\ z\end{bmatrix}-\begin{bmatrix}1\\ 0\end{bmatrix}K\begin{bmatrix}x_{11}\\ z\end{bmatrix}.$$

Choosing the eigenvalues $\lambda_{1,2}=-1$ we obtain $G_{11}=2$ and $G_2=1$ which also determines $G_{12}=1$.

Applying the control law (6) we require $\rho>\phi^{\sup}\geq\lVert G\phi\rVert$.

B. Conventional ISMC design

For comparison we design two nominal controls. Each of them is augmented by a sliding-mode control to compensate the matched uncertainties.

1) Nominal PI state-feedback controller: A PI state-feedback control law is suitable to compensate stationary uncertainties, choosing the integrator state v with $\dot v=x_{11}$. The nominal control law is then given by

$$u_0(x)=\frac{1}{b_2(t,x)}\left(K\begin{pmatrix}x\\ v\end{pmatrix}-f_2(t,x)\right) \tag{32}$$

with $K\in\mathbb{R}^{1\times 4}$. The state-feedback matrix K is to be designed for the closed loop system

$$\begin{bmatrix}\dot x_{11}\\ \dot x_{12}\\ \dot x_2\\ \dot v\end{bmatrix}=\begin{bmatrix}0&1&1&0\\0&0&1&0\\0&0&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}x_{11}\\ x_{12}\\ x_2\\ v\end{bmatrix}+\begin{bmatrix}0\\0\\1\\0\end{bmatrix}K\begin{bmatrix}x_{11}\\ x_{12}\\ x_2\\ v\end{bmatrix}.$$

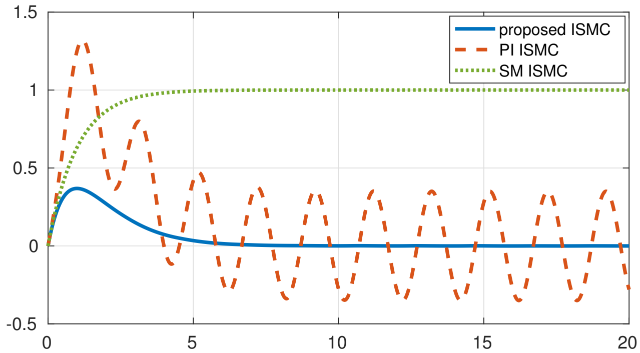

Fig. 1. Desired output $h(x) = x_{11}$.

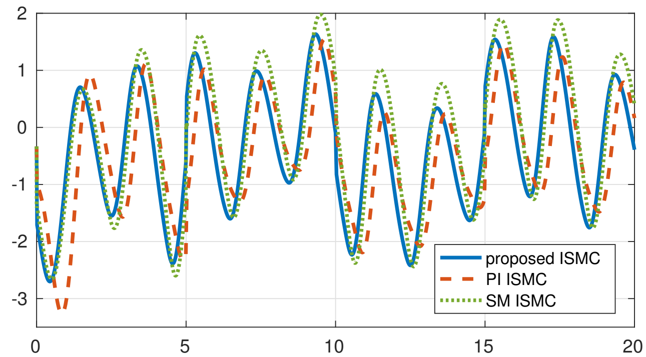

Fig. 2. Control variable u.

Choosing all four eigenvalues as -1 yields

$$K = \begin{bmatrix} -3 & -3 & -4 & -1 \end{bmatrix}.$$

2) Nominal sliding-mode control: The decoupling condition (28) may also be used to design a switching function $s^{\mathrm{SM}} = G^{\mathrm{SM}} x$ with $G^{\mathrm{SM}} := \begin{bmatrix} G_{11}^{\mathrm{SM}} & G_{12}^{\mathrm{SM}} & G_2^{\mathrm{SM}} \end{bmatrix}$ for a nominal sliding-mode controller.

For the reduced dynamics $s^{\mathrm{SM}} \equiv 0$ we obtain:

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} = \begin{bmatrix} -\frac{G_{11}^{\mathrm{SM}}}{G_2^{\mathrm{SM}}} & 1 - \frac{G_{12}^{\mathrm{SM}}}{G_2^{\mathrm{SM}}} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}.$$

Thus, decoupling is obtained for $G_{12}^{\mathrm{SM}} = G_2^{\mathrm{SM}}$. Choosing the remaining eigenvalue as -1, we get $G^{\mathrm{SM}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$. With $\rho^{\mathrm{SM}} > \|G^{\mathrm{SM}} \phi\|$ the sliding-mode control reads

$$u_0(x) = \frac{-1}{G^{\mathrm{SM}} b(t,x)} \left(G^{\mathrm{SM}} f(t,x) + \rho^{\mathrm{SM}} \operatorname{sgn}(s^{\mathrm{SM}}) \right), \quad (33)$$

where $b(t,x) = \begin{bmatrix} 0 & 0 & b_2(t,x) \end{bmatrix}^\top$.

3) ISMC law: For compensating the matched uncertainty ϕ_2 we pick the sliding manifold as $s^{\mathrm{ISMC}} = x_2 + z$ with integrator state z as in (18). The complete ISM control is

$$u = u_0 - \rho^{\mathrm{ISMC}} \operatorname{sgn}\left(s^{\mathrm{ISMC}}\right),$$

with $\rho^{\mathrm{ISMC}} > \|\phi_2\|$ and nominal control u_0 in (32) or (33).

C. Simulation Results

We simulate the system (31) with the disturbances

$$\begin{aligned} \phi_{11}(x_{11}) &= 1, \\ \phi_{12}(t, x_{11}, x_{12}, x_2) &= 4\sin(\pi t), \\ \phi_2(t, x_{11}, x_{12}, x_2) &= \begin{cases} 2 & \text{for } 5(2k) \le t < 5(2k+1) \\ 0 & \text{for } 5(2k+1) \le t < 5(2(k+1)) \end{cases} \end{aligned}$$

with $k \in \mathbb{N}$. The systems are initialised at the origin and the sliding-mode gain is chosen as $\rho^{\mathrm{SM}} = \rho = 10$ to dominate the uncertainty in each case. In order to avoid chattering we approximate $\operatorname{sgn}(s) \approx \frac{s}{|s|+\varepsilon}$ with $\varepsilon = 0.001$.

Note, that the reduced dynamics of the PI-ISCM is of fourth order, whereas the SM-ISCM and our proposed ISCM have reduced dynamics of third order.

Fig. 1 shows the evolution of the desired output $h(x) = x_{11}$. Under the proposed control law, the matched disturbance ϕ_2 as well as the unmatched disturbance ϕ_{12} do not influence the course of the state at all. The unmatched uncertainty ϕ_{11} influences the transient behaviour of the state but is completely compensated by the integrator for $t \to \infty$. The PI-ISMC compensates the matched disturbance ϕ_2 and the stationary influence of ϕ_{11}, but the unmatched disturbance ϕ_{12} shows a strong impact. The ISMC with SM nominal control compensates the matched uncertainty ϕ_2 also decouples the unmatched uncertainty ϕ_{12}, but the unmatched uncertainty ϕ_{11} causes a stationary error.

Fig. 2 shows the control variable of the three control laws. The matched uncertainty ϕ_2 is compensated by all three control laws which causes the discontinuity of the control variables at 5, 10 and 15 time units. The unmatched uncertainty ϕ_{11} results in an offset of the control variable of the SM-ISMC compared to the other two control laws.

VI. Conclusion

We have presented a novel and systematic design method for an integral sliding mode controller for nonlinear systems in regular form. The approach is capable of compensating matched and also a class of unmatched uncertainties. Moreover, we derive conditions that allow for complete decoupling of a class of unmatched uncertainties. The design is focused on the direct choice of output for which stationary accuracy is required. The resulting controller can be cast into the standard framework of integral sliding mode approaches and a direct comparison of the approaches is provided. Our proposed approach integrates the nominal control and the sliding manifold into one design procedure. Whenever integral action is included in the nominal control of the conventional design approach our proposed design result in dynamics of lower order. For linear systems, the stability and decoupling conditions enable a very simple design procedure that is readily applicable. A simulation example illustrates the design and highlights the compensation and decoupling properties in comparison to conventional ISMC.

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